2. Factorial Designs

- Analysis of variance of a factorial experiments
- Two-level factorial designs
- Factor effects estimation
- Assessing the effects’ significance
- Normal plot of effects

Analysis of Variance of Factorial Design

Yield in Paper Industry

Problem: Percentage yield as a function of reaction time, temperature and pressure in a paper industry.

Factor Levels:

<table>
<thead>
<tr>
<th>Levels</th>
<th>Factor 1</th>
<th>Factor 2</th>
<th>Factor 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (hrs.)</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Temp (°C)</td>
<td>30</td>
<td>70</td>
<td>100</td>
</tr>
<tr>
<td>Pressure (psi)</td>
<td>30</td>
<td>70</td>
<td>100</td>
</tr>
</tbody>
</table>

Data Set:

<table>
<thead>
<tr>
<th>obs</th>
<th>time</th>
<th>Temp</th>
<th>Press</th>
<th>yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>68.5</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>74.5</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>70.5</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>72.8</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>72.0</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>69.5</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>72.5</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>75.5</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>76.0</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>73.0</td>
</tr>
<tr>
<td>11</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>75.0</td>
</tr>
<tr>
<td>12</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>72.5</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>80.1</td>
</tr>
<tr>
<td>14</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>81.5</td>
</tr>
</tbody>
</table>

Analysis of Variance of Factorial Design

Yield in Paper Industry

If we now include in the model the factor time ($A$) and all second order interactions.

Model:

$$y = \mu + \tau_A + \tau_B + \tau_C + \tau_{AB} + \tau_{AC} + \tau_{BC} + \epsilon$$

ANOVA Table:

<table>
<thead>
<tr>
<th>Df</th>
<th>Sum Sq</th>
<th>Mean Sq</th>
<th>F value</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>time</td>
<td>2</td>
<td>42.112</td>
<td>21.056</td>
<td>0.0096484</td>
</tr>
<tr>
<td>Temp</td>
<td>2</td>
<td>110.732</td>
<td>55.366</td>
<td>0.0004788</td>
</tr>
<tr>
<td>Press</td>
<td>2</td>
<td>68.136</td>
<td>34.068</td>
<td>0.0023445</td>
</tr>
<tr>
<td>time:Temp</td>
<td>4</td>
<td>35.184</td>
<td>8.796</td>
<td>0.0558947</td>
</tr>
<tr>
<td>time:Press</td>
<td>4</td>
<td>136.437</td>
<td>34.109</td>
<td>0.0010480</td>
</tr>
<tr>
<td>Residuals</td>
<td>8</td>
<td>19.223</td>
<td>2.403</td>
<td></td>
</tr>
</tbody>
</table>

Conclusion: Main factor effects and second order interactions are significant.

Analysis of Variance of Factorial Design

Yield in Paper Industry

Finally, if we decompose further the sum of squares of the residuals by including third order interactions. The resulting model is

$$y = \mu + \tau_A + \tau_B + \tau_C + \tau_{AB} + \tau_{AC} + \tau_{BC} + \tau_{ABC} + \epsilon$$

ANOVA Table:

<table>
<thead>
<tr>
<th>Df</th>
<th>Sum Sq</th>
<th>Mean Sq</th>
<th>F value</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>time</td>
<td>2</td>
<td>42.112</td>
<td>21.056</td>
<td>0.0096484</td>
</tr>
<tr>
<td>Temp</td>
<td>2</td>
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<td>55.366</td>
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<tr>
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</tr>
<tr>
<td>time:Press</td>
<td>4</td>
<td>136.437</td>
<td>34.109</td>
<td>0.0010480</td>
</tr>
<tr>
<td>Residuals</td>
<td>8</td>
<td>0.000</td>
<td>0.000</td>
<td></td>
</tr>
</tbody>
</table>

Conclusion: All main, second and third order interactions are estimable but there are no degrees of freedom left to estimate the error variance. Thus, it is not possible to assess statistically the significance of such effects.

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Two-Level Factorial Designs

To perform a factorial design, you select a fixed number of levels of each of a number of factors (variables) and then run experiments in all possible combinations. The factors can be quantitative (e.g., various temperatures) or qualitative (e.g., different methods).

Two-level factorial designs are of special importance because of the following:

1. They require relatively few runs per factor studied.
2. The interpretation of the observations produced by the designs can proceed largely by using common sense and computer graphics.
3. When the factors are quantitative, although unable to fully explore a wide region in the factor space, they often determine a promising direction for further experimentation.
4. Design can be suitably augmented when a more thorough local exploration is needed—a process called sequential assembly.
5. The form the basis for two-level fractional factorial designs. Particularly useful for factor screening and building blocks in the sequential assembly of experimental designs.

Two-Level Factorial Designs
Pilot Plant Investigation

2\(^3\) Full Factorial Design

The next table shows a 2\(^3\) factorial design with 2 quantitative factors, temperature \(T\) and concentration \(C\), and one qualitative factor, catalyst \(K\). The response \(y\) is the chemical yield.

<table>
<thead>
<tr>
<th>Factor Levels</th>
<th>Factors Response</th>
<th>run (T) (C) (K)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(T) (C) (K)</td>
<td>run</td>
<td>(y)</td>
</tr>
<tr>
<td>Factor</td>
<td>label low (-) high (+)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Temperature (°C)</td>
<td>160 180</td>
<td>1</td>
<td>60</td>
</tr>
<tr>
<td>Concentration (%)</td>
<td>20 40</td>
<td>2</td>
<td>72</td>
</tr>
<tr>
<td>Catalyst (type)</td>
<td>A B</td>
<td>3</td>
<td>54</td>
</tr>
<tr>
<td></td>
<td>4 ++</td>
<td>4</td>
<td>68</td>
</tr>
<tr>
<td></td>
<td>5 -- +</td>
<td>5</td>
<td>52</td>
</tr>
<tr>
<td></td>
<td>6 ++</td>
<td>6</td>
<td>83</td>
</tr>
<tr>
<td></td>
<td>7 -- +</td>
<td>7</td>
<td>45</td>
</tr>
<tr>
<td></td>
<td>8 + + +</td>
<td>8</td>
<td>80</td>
</tr>
</tbody>
</table>

Temperature \(T\) main effect:

\[
\tau_T = \frac{\bar{y}_{T+} - \bar{y}_{T-}}{4} = \frac{72 + 68 + 83 + 80}{4} = \frac{60 + 54 + 52 + 45}{4} = 75.75 - 52.75 = 23
\]

\[
\tau_T = \sum \frac{C_r d_i}{n/2}
\]

\footnote{BHH52, Chapter 5; BHH Chapter 10.}
Two-Level Factorial Designs
Pilot Plant Investigation
Second Order Interaction Effects:

Temperature–Catalyst Interaction Effect $TK: T \times K = T \ast K = T : K$

Using the $+/-$ coding:

\[ c_{TK} = c_T \ast c_K \]

Interaction Effect

\[ \tau_{TK} = \frac{\sum c_{TK}y_i}{n/2} = 10 \]

Temperature–Concentration–Catalyst Interaction Effect $TCK: T \times K = T \ast C \ast K = T : C \ast K$

Using the $+/-$ coding:

\[ c_{TCK} = c_T \ast c_C \ast c_K \]

Interaction Effect

\[ \tau_{TCK} = \frac{\sum c_{TCK}y_i}{n/2} = 10 \]

Two-Level Factorial Designs
Pilot Plant Investigation
Third Order Interaction Effects:

Temperature–Concentration–Catalyst Interaction Effect $TCK: T \times K = T \ast C \ast K = T : C \ast K$

Using the $+/-$ coding:

\[ c_{TCK} = c_T \ast c_C \ast c_K \]

Interaction Effect

\[ \tau_{TCK} = \frac{\sum c_{TCK}y_i}{n/2} = 10 \]
Two-Level Factorial Designs

Pilot Plant Investigation

Normal Plot of Effects:

![Normal Plot of Effects](image)

Two-Level Factorial Designs

Process Development Study

Contrast coefficients

\[
\begin{align*}
&x_1, x_2, x_3, x_4, x_12, x_13, x_14, x_23, x_24, x_34, x_{123}, x_{124}, x_{134}, x_{1234} \\
&\text{conversion}
\end{align*}
\]

\[
\begin{align*}
1 & -1 -1 -1 -1 1 1 1 1 -1 -1 -1 1 1 & 70 \\
1 & -1 -1 -1 -1 1 1 1 1 -1 -1 -1 1 1 & 69 \\
1 & -1 -1 -1 -1 1 1 1 1 -1 -1 -1 1 1 & 89 \\
1 & -1 -1 -1 -1 1 1 1 1 -1 -1 -1 1 1 & 81 \\
1 & -1 -1 -1 -1 1 1 1 1 -1 -1 -1 1 1 & 62 \\
1 & -1 -1 -1 -1 1 1 1 1 -1 -1 -1 1 1 & 88 \\
1 & -1 -1 -1 -1 1 1 1 1 -1 -1 -1 1 1 & 81 \\
1 & -1 -1 -1 -1 1 1 1 1 -1 -1 -1 1 1 & 60 \\
1 & -1 -1 -1 -1 1 1 1 1 -1 -1 -1 1 1 & 49 \\
1 & -1 -1 -1 -1 1 1 1 1 -1 -1 -1 1 1 & 88 \\
1 & -1 -1 -1 -1 1 1 1 1 -1 -1 -1 1 1 & 82 \\
1 & -1 -1 -1 -1 1 1 1 1 -1 -1 -1 1 1 & 60 \\
1 & -1 -1 -1 -1 1 1 1 1 -1 -1 -1 1 1 & 52 \\
1 & -1 -1 -1 -1 1 1 1 1 -1 -1 -1 1 1 & 86 \\
1 & -1 -1 -1 -1 1 1 1 1 -1 -1 -1 1 1 & 79
\end{align*}
\]

Assuming third and higher order interactions are negligible:

\[
\sigma^2 = \frac{1}{N} \left[ \left(-0.75\right)^2 + \left(+0.50\right)^2 + \left(-0.25\right)^2 + \left(-0.75\right)^2 + \left(-0.75\right)^2 \right] = 0.30
\]

\[
se = \sqrt{0.30} = 0.55
\]

E. Barrios  Design and Analysis of Engineering Experiments 2–13

E. Barrios  Design and Analysis of Engineering Experiments 2–15

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\[\text{BHH2e, Chapter 5; BHH Chapter 10.}\]
Two-Level Factorial Designs

Process Development Study

Normal Plot of Effects

Normal Plot

Normal Probability Density

Normal Plot Cumulative Distribution

Normal Plot Paper
Normal Plot

(a) Normal probability density

(b) Normal probability distribution

(c) Normal probability plot

Simulated Normal Samples

Process Development Study

Normal Probability Paper

order effect factor Prob

Prob = 100(i - 1/2)/15
Two-Level Factorial Designs
Process Development Study

Minitab Normal Plot of Effects

Normal Probability Plot of the Effects
(response is y, alpha = .05)

- Effect Type
  - Not Significant
  - Significant

- Factor Name
  - A
  - B
  - C
  - D

Lenth's PSE = 0.75

Two-Level Factorial Designs
Normal Plot of Effects